NILPOTENT COMPLETIONS OF GROUPS, GROTHENDIECK PAIRS, AND FOUR PROBLEMS OF BAUMSLAG

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Abstract. Two groups are said to have the same nilpotent genus if they have the same nilpotent quotients. We answer four questions of Baumslag concerning nilpotent completions. (i) There exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one is finitely presented and the other is not. (ii) There exists a pair of finitely presented, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has a solvable conjugacy problem and the other does not. (iii) There exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has finitely generated second homology $H_2(-,\mathbb{Z})$ and the other does not. (iv) A non-trivial normal subgroup of infinite index in a finitely generated parafree group cannot be finitely generated.

1. Introduction

If each finite subset of a group Γ injects into some nilpotent (or finite) quotient of Γ, then it is reasonable to expect that one will be able to detect many properties of Γ from the totality of its nilpotent (or finite) quotients. Attempts to lend precision to this observation, and to test its limitations, have surfaced repeatedly in the study of discrete and profinite groups over the last forty years, and there has been a particular resurgence of interest recently, marked by several notable breakthroughs. We take up this theme here, with a focus on the nilpotent completions of residually torsion-free-nilpotent groups.

We begin by recalling some terminology. A group Γ is said to be *residually nilpotent* (resp. *residually torsion-free-nilpotent*) if for each non-trivial $\gamma \in \Gamma$ there exists a nilpotent group (resp. torsion-free-nilpotent group) Q and a homomorphism $\phi : \Gamma \to Q$ with $\phi(\gamma) \neq 1$. Thus Γ is residually nilpotent if and only $\bigcap \Gamma_n = 1$, where Γ_n, the n-th term of the *lower central series* of Γ, defined inductively by setting $\Gamma_1 = \Gamma$ and defining $\Gamma_{n+1} = \langle [x,y] : x \in \Gamma_n, y \in \Gamma \rangle$.

We say that two residually nilpotent groups Γ and Λ have the same *nilpotent genus*[1] if they have the same lower central series quotients; i.e. $\Gamma/\Gamma_c \cong \Lambda/\Lambda_c$ for all $c \geq 1$.

Residually nilpotent groups with the same nilpotent genus as a free group are termed *parafree*. In [4] Gilbert Baumslag surveyed the state of the art concerning groups of the same nilpotent genus with particular emphasis on the nature of parafree groups. He concludes by listing a number of open problems that are of particular importance in the field. We will address Problems 2, 4, and 6 from his list [4], two of which he raised again in [5] where he emphasised the importance of the third of the problems described below.

Problem 1.1. *Does there exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one is finitely presented and the other is not?*

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[1]Baumslag uses the simpler term "genus", but this conflicts with the usage of the term in the study of profinite groups, and since we study different completions it seems best to be more precise here.

Problem 1.2. *Does there exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has a solvable conjugacy problem and the other does not?*

Problem 1.3. *Does there exists a pair of finitely generated, residually torsion-free-nilpotent groups of the same nilpotent genus such that one has finitely generated second homology $H_2(-,\mathbb{Z})$ and the other does not?*

Problem 1.4. *Let G be a finitely generated parafree group and let $N < G$ be a finitely generated, non-trivial, normal subgroup. Must N be of finite index in G?*

In [4] Baumslag gives some evidence to suggest that pairs of groups as in Problem 1.2 might exist; he had earlier proved that there exist finitely presented residually torsion-free nilpotent groups with unsolvable conjugacy problem [3]. Baumslag also proves a partial result in connection with Problem 1.4 (see Theorem 7 of [4]). Note that Problem 1.4 is well known to have a positive answer for free groups.

In this note we answer these questions. Concerning the possible divergence in behaviour between groups in the same nilpotent genus, we prove:

Theorem A.
(1) *There exist pairs of finitely generated, residually torsion-free-nilpotent groups $H \hookrightarrow D$ of the same nilpotent genus such that D is finitely presented and H is not.*
(2) *There exist pairs of finitely presented, residually torsion-free-nilpotent groups $P \hookrightarrow \Gamma$ of the same nilpotent genus such that Γ has a solvable conjugacy problem and P does not.*
(3) *There exist pairs of finitely generated, residually torsion-free-nilpotent groups $N \hookrightarrow \Gamma$ of the same nilpotent genus such that $H_2(\Gamma,\mathbb{Z})$ is finitely generated but $H_2(N,\mathbb{Z})$ is not.*

In the above theorem, the pairs of groups that we construct have the same profinite completions. The following result strengthens items (1) and (3) of the above theorem.

Theorem B. *There exist pairs of finitely generated residually torsion-free-nilpotent groups $N \hookrightarrow \Gamma$ that have the same nilpotent genus and the same profinite completion, but Γ is finitely presented while $H_2(N,\mathbb{Q})$ is infinite dimensional.*

The preceding results emphasise how divergent the behaviour can be within a nilpotent genus. Our solution to Problem 1.4, in contrast, establishes a commonality among parafree groups.

Theorem C. *Let G be a finitely generated parafree group, and let $N < G$ be a non-trivial normal subgroup. If N is finitely generated, then G/N is finite.*

This paper is organised as follows. In Section 2 we recall some basic properties about profinite and pro-nilpotent completions of discrete groups and about the correspondence between the subgroup structure of the discrete group and that of its various completions. In Section 3 we observe that if a map of finitely generated discrete groups $P \hookrightarrow \Gamma$ induces an isomorphism of profinite completions, then P and Γ have the same nilpotent genus. (If ones assumes merely that $\widehat{P} \cong \widehat{\Gamma}$, then the genus need not be the same.) Our proof of Theorem A involves the construction of carefully crafted pairs of residually torsion-free-nilpotent groups $u : P \hookrightarrow \Gamma$ such that $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism. Theorem A(1) is proved in Section 4, and Theorem A(2) is proved in Section 5. The proof of Theorem B is more elaborate: it involves the construction of a finitely presented group with particular properties that may be of independent interest (Proposition 6.5).

Theorem C is proved in Section 7. As with our other results, the proof exploits the theory of profinite groups. It also relies on results concerning the L^2-Betti numbers of discrete groups. A key observation here is that the first L^2-Betti number of a finitely presented residually torsion-free-nilpotent group is an invariant of its pro-p completion for an arbitrary prime p (see Corollary 7.6). Theorem C will emerge as a special case of a result concerning L^2-Betti numbers for dense subgroups of free pro-p groups (Theorem 7.7). In Section 8 we discuss the implications of our results with regard to lattices in connected Lie groups and other groups of geometric interest.

In relation to Thoerem A(1), we draw the reader's attention to a recent paper of Alexander Lubotzky [29] in which he constructs pairs of groups that have isomorphic profinite completions but have finiteness lengths that can be chosen arbitrarily. These examples are S-arithmetic groups over function fields; they do not have infinite proper quotients and therefore are not residually torsion-free nilpotent, but they are residually finite-nilpotent.

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2. Profinite and Pro-Nilpotent Completions

Let Γ be a finitely generated group. If one orders the normal subgroups of finite index $N < \Gamma$ by reverse inclusion, then the quotients Γ/N form an inverse system whose inverse limit

$$\widehat{\Gamma} = \varprojlim \Gamma/N$$

is the *profinite completion* of Γ. Similarly, the *pro-nilpotent completion*, denoted $\widehat{\Gamma}_{\rm nil}$ is the inverse limit of the nilpotent quotients of Γ, and the *pro-(finite nilpotent)* completion, denoted $\widehat{\Gamma}_{\rm fn}$, is the inverse limit of the finite nilpotent quotients of Γ. Given a prime p, the the *pro-p completion* $\widehat{\Gamma}_p$ is the inverse limit of the finite p-group quotients of Γ.

Remarks 2.1.
(1) The inverse limit topology on $\widehat{\Gamma}$, on $\widehat{\Gamma}_p$, and on $\widehat{\Gamma}_{\rm fn}$ makes them *compact* topological groups. The induced topologies on Γ are called the profinite, pro-p and *pro-(finite nilpotent)* topologies, respectively.
(2) We do not view $\widehat{\Gamma}_{\rm nil}$ as a topological group, and the absence of a useful topology makes it a less interesting object.
(3) By construction, $\widehat{\Gamma}$ (resp. $\widehat{\Gamma}_{\rm nil}$) is residually finite (resp. nilpotent), while $\widehat{\Gamma}_{\rm fn}$ is residually finite-nilpotent, and $\widehat{\Gamma}_p$ is residually p.
(4) The natural homomorphism $\Gamma \to \widehat{\Gamma}$ is injective if and only if Γ is residually finite, while the natural maps $\Gamma \to \widehat{\Gamma}_{\rm nil}$ and $\Gamma \to \widehat{\Gamma}_{\rm fn}$ are injective if and only if Γ is residually nilpotent.
(5) $\Gamma \to \widehat{\Gamma}_{\rm nil}$ is an isomorphism if and only if Γ is nilpotent.
(6) The quotients Γ/Γ_c of Γ by the terms of its lower central series are cofinal in the system of all nilpotent quotients, so one can equally define $\widehat{\Gamma}_{\rm nil}$ to be the inverse limit of these.
(7) If $\widehat{\Gamma}_{\rm nil} \cong \widehat{\Lambda}_{\rm nil}$ then $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$, but the converse is false in general since there exist finitely generated non-isomorphic nilpotent groups that have the same finite quotients [38].
(8) Less obviously, if Γ and Λ are finitely generated and have the same nilpotent genus then $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$. To see this note that having the same nilpotent genus is equivalent to the statement that Γ and Λ have the same nilpotent quotients, in particular the same finite nilpotent quotients. The conclusion $\widehat{\Gamma}_{\rm fn} \cong \widehat{\Lambda}_{\rm fn}$ follows from Theorem 3.2.7 of [39].
(9) Finite p-groups are nilpotent, so for every prime p there is a natural epimorphism $\widehat{\Gamma}_{\rm fn} \twoheadrightarrow \widehat{\Gamma}_p$. It follows that finitely generated groups of the same nilpotent genus have isomorphic pro-p completions for all primes p.
(10) Every homomorphism of discrete groups $u : \Gamma \to \Lambda$ induces maps $\widehat{u} : \widehat{\Gamma} \to \widehat{\Lambda}$ and $\widehat{u}_{(p)} : \widehat{\Gamma}_p \to \widehat{\Lambda}_p$ and $\widehat{u}_{\rm nil} : \widehat{\Gamma}_{\rm nil} \to \widehat{\Lambda}_{\rm nil}$ and $\widehat{u}_{\rm fn} : \widehat{\Gamma}_{\rm fn} \to \widehat{\Lambda}_{\rm fn}$.

The image of the canonical map $\Gamma \to \widehat{\Gamma}$ is dense regardless of whether Γ is residually finite or not, so the restriction to Γ of any continuous epimomorphism from $\widehat{\Gamma}$ to a finite group is onto. A deep theorem of Nikolov and Segal [36] implies that if Γ is finitely generated then *every* homomorphism from $\widehat{\Gamma}$ to a finite group is continuous. And the universal property of $\widehat{\Gamma}$ ensures that every homomorphism from Γ to a finite group extends uniquely to $\widehat{\Gamma}$. Thus we have the following basic

result in which $\mathrm{Hom}(\Gamma, Q)$ denotes the set of homomorphisms from the group Γ to the group Q, and $\mathrm{Epi}(\Gamma, Q)$ denotes the set of epimorphisms.

Lemma 2.2. *Let Γ be a finitely generated group and let $\iota : \Gamma \to \widehat{\Gamma}$ be the natural map to its profinite completion. Then, for every finite group Q, the map $\mathrm{Hom}(\widehat{\Gamma}, Q) \to \mathrm{Hom}(\Gamma, Q)$ defined by $g \mapsto g \circ \iota$ is a bijection, and this restricts to a bijection $\mathrm{Epi}(\widehat{\Gamma}, Q) \to \mathrm{Epi}(\Gamma, Q)$.*

If one replaces $\widehat{\Gamma}$ by $\widehat{\Gamma}_{\mathrm{nil}}$ or $\widehat{\Gamma}_{\mathrm{fn}}$, one obtains bijections for finite nilpotent groups Q. And for $\widehat{\Gamma}_p$ one obtains bijections when Q is a finite p-group.

Closely related to this, we have the following basic but important fact relating the subgroup structures of Γ and $\widehat{\Gamma}_{\mathrm{fn}}$ and $\widehat{\Gamma}_p$ (see [39] Proposition 3.2.2, and note that the argument is valid for other profinite completions).

Notation. Given a subset X of a pro-finite group G, we write $\overline{X}$ to denote the closure of X in G.

Proposition 2.3. *Let $\mathcal{C}$ be the class of finite nilpotent groups or finite p-groups (for a fixed prime p). If Γ is a finitely generated discrete group which is residually $\mathcal{C}$, then there is a one-to-one correspondence between the set $\mathcal{X}$ of subgroups of Γ that are open in the pro-$\mathcal{C}$ topology on Γ, and the set $\mathcal{Y}$ of all open subgroups in the pro-$\mathcal{C}$ completion of Γ. Identifying Γ with its image in the completion, this correspondence is given by:*

- *For $H \in \mathcal{X}$, $H \mapsto \overline{H}$.*
- *For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.*

If $H, K \in \mathcal{X}$ and $K < H$ then $[H : K] = [\overline{H} : \overline{K}]$. Moreover, $K \lhd H$ if and only if $\overline{K} \lhd \overline{H}$, and $\overline{H}/\overline{K} \cong H/K$.

Remark 2.4. It is proved [2] that for a finitely generated group Γ, *every* subgroup of finite index in $\widehat{\Gamma}_{\mathrm{fn}}$ is open, and likewise in $\widehat{\Gamma}_p$. (See [36] for the case of a general profinite group.) Thus every normal subgroup of index d in $\widehat{\Gamma}_{\mathrm{fn}}$ is the closure of a subgroup $K \lhd \Gamma$ such that Γ/K is nilpotent and $|\Gamma/K| = d$, and every subgroup of index $d = p^k$ in $\widehat{\Gamma}_p$ is the closure of a subgroup of index p^k in Γ.

Corollary 2.5. *Let Γ be a finitely generated group and for each $d \in \mathbb{N}$ let $M(d)$ denote the intersection of all the normal subgroups $\Delta \lhd \Gamma$ of index $\leq d$ such that Γ/Δ is nilpotent. Let $\overline{M(d)}$ be the closure of $M(d)$ in $\widehat{\Gamma}_{\mathrm{fn}}$. Then $\overline{M(d)}$ is the intersection of all the normal subgroups of index $\leq d$ in $\widehat{\Gamma}_{\mathrm{fn}}$, and hence $\bigcap_d \overline{M(d)} = 1$.*

Proof. In the light of Remark 2.4, it suffices to show that if $K_1, K_2 < \Gamma$ are normal and $Q_1 = \Gamma/K_1$ and $Q_2 = \Gamma/K_2$ are finite and nilpotent, then $\overline{K_1 \cap K_2} = \overline{K_1} \cap \overline{K_2}$. But $\overline{K_1 \cap K_2}$ is the kernel of the extension of $\Gamma \to Q_1 \times Q_2$ to $\widehat{\Gamma}_{\mathrm{fn}}$, while $\overline{K}_1 \times \overline{K}_2$ is the kernel of the map $\widehat{\Gamma}_{\mathrm{fn}} \to Q_1 \times Q_2$ that one gets by extending each of $\Gamma \to Q_i$ and then taking the direct product. These maps coincide on Γ, which is dense, and are therefore equal. $\square$

The same argument establishes:

Corollary 2.6. *Let p be a prime and let Γ be a finitely generated group. For each $d \in \mathbb{N}$ let $N(d)$ denote the intersection of all the normal subgroups $\Delta \lhd \Gamma$ of p-power index less than d. Let $\overline{N(d)}$ be the closure of $N(d)$ in $\widehat{\Gamma}_{\mathrm{fn}}$. Then $\overline{N(d)}$ is the intersection of all the normal subgroups of index $\leq d$ in $\widehat{\Gamma}_{\mathrm{fn}}$, and hence $\bigcap_d \overline{N(d)} = 1$.*

Remark 2.7. A key feature of the subgroups $M(d)$ [resp. $N(d)$] is that they form a fundamental system of open neighbourhoods of $1 \in \Gamma$ defining the pro-(finite nilpotent) [resp. pro-p] topology. If we had merely taken an exhausting sequence of normal subgroups in Γ, then we would not have been able to conclude that the intersection in $\widehat{\Gamma}_{\mathrm{fn}}$ [resp. $\widehat{\Gamma}_p$] of their closures was trivial.

2.1. **Subgroups of p-Power Index.** A key advantage of the class of p-groups over the class of finite nilpotent groups is that the former is closed under extensions whereas the latter is not. The importance of this from our point of view is that it means that the induced topology on normal subgroups of p-power index $\Lambda < \Gamma$ behaves well. The following is a consequence of Lemma 3.1.4(a) of [39] (in the case where $\mathcal{C}$ is the class of finite p-groups).

Lemma 2.8. *Let p be a prime and let Γ be a finitely generated group that is residually-p. If $\Lambda < \Gamma$ is a normal subgroup of p-power index, then the natural map $\widehat{\Lambda}_p \to \overline{\Lambda} < \widehat{\Gamma}_p$ is an isomorphism.*

2.2. **Betti numbers.** The first Betti number of a finitely generated group is

$$b_1(\Gamma) = \dim_{\mathbb{Q}} \, (\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Given any prime p, one can detect $b_1(\Gamma)$ in the p-group quotients of Γ, since it is the *least integer b such that Γ surjects $(\mathbb{Z}/p^k\mathbb{Z})^b$ for for every $k \in \mathbb{N}$.* We exploit this observation as follows:

Lemma 2.9. *Let Λ and Γ be finitely generated groups and let p be a prime. If Λ is isomorphic to a dense subgroup of $\widehat{\Gamma}_p$, then $b_1(\Lambda) \geq b_1(\Gamma)$.*

Proof. For every finite p-group A, each epimorphism $\widehat{\Gamma} \to A$ will restrict to an epimorphism on both Γ and Λ (since by density Λ cannot be contained in a proper closed subgroup). But the resulting map $\mathrm{Epi}(\widehat{\Gamma}, A) \to \mathrm{Epi}(\Lambda, A)$ need not be surjective, in contrast to Lemma 2.2. Thus if Γ surjects $(\mathbb{Z}/p^k\mathbb{Z})^b$ then so does Λ (but perhaps not vice versa). □

Corollary 2.10. *If Λ and Γ are finitely generated and $\widehat{\Lambda}_p \cong \widehat{\Gamma}_p$, then $b_1(\Lambda) = b_1(\Gamma)$.*

3. Grothendieck Pairs and Pro-Nilpotent completions

Let Γ be a residually finite group and let $u : P \hookrightarrow \Gamma$ be the inclusion of a subgroup P. Then (Γ, P) is called a *Grothendieck Pair* if the induced homomorphism $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism (and u is not). Grothendieck [22] asked about the existence of such pairs of finitely presented groups and the first such pairs were constructed by Bridson and Grunewald in [14]. The analogous problem for finitely generated groups had been settled earlier by Platonov and Tavgen [37]. Both constructions rely on versions of the following result (cf. [37], [14] Theorem 5.2 and [11]).

We remind the reader that the *fibre product* $P < H \times H$ associated to an epimorphism of groups $p : H \twoheadrightarrow Q$ is the subgroup $P = \{(x, y) \mid p(x) = p(y)\}$.

Proposition 3.1. *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups with Γ finitely generated, and let P be the associated fibre product. Suppose that $Q \neq 1$ is finitely presented, has no proper subgroups of finite index, and $H_2(Q, \mathbb{Z}) = 0$. Then*

1. *$(\Gamma \times \Gamma, P)$ is a Grothendieck pair;*
2. *if N is finitely generated then (Γ, N) is a Grothendieck pair.*

We shall exploit variations on these constructions in answering Problems 1.1 and 1.2. The link between Baumslag's problems and Grothendieck's problem is explained by the following proposition.

Proposition 3.2. *Let $u : P \hookrightarrow \Gamma$ be a pair of finitely generated, residually finite groups, and for each $c \geq 1$, let $u_c : P/P_c \to \Gamma/\Gamma_c$ be the induced homomorphism. If (Γ, P) is a Grothendieck Pair, then u_c is an isomorphism for all $c \geq 1$, and hence $\widehat{u}_{\mathrm{nil}} : \widehat{P}_{\mathrm{nil}} \to \widehat{\Gamma}_{\mathrm{nil}}$ is an isomorphism. In particular, if P and Γ are residually nilpotent, then they have the same nilpotent genus.*

Remark 3.3. In this proposition, it is vital that the isomorphism between $\widehat{P}$ and $\widehat{\Gamma}$ is induced by a map $P \to \Gamma$ of discrete groups. For example, as we remarked earlier, there exist finitely generated nilpotent groups that are not isomorphic but have the same profinite completion. A nilpotent group is its own pro-nilpotent completion, so these examples have the same profinite genus but different nilpotent genera.

Lemma 3.4. *Let $u : P \hookrightarrow \Gamma$ be a pair of finitely generated, residually finite groups. If (Γ, P) is a Grothendieck Pair, then for every finite group G, the map $q \mapsto q \circ u$ defines a bijection* $\mathrm{Epi}(\Gamma, G) \to \mathrm{Epi}(P, G)$.

Proof. As in Lemma 2.2, by restricting homomorphisms $\widehat{\Gamma} \to G$ to $\Gamma < \widehat{\Gamma}$ we obtain a bijection $\mathrm{Epi}(\widehat{\Gamma}, G) \to \mathrm{Epi}(\Gamma, G)$. Similarly, there is a bijection $\mathrm{Epi}(\widehat{P}, G) \to \mathrm{Epi}(P, G)$. And the isomorphism $\widehat{u}$ induces a bijection $\mathrm{Epi}(\widehat{\Gamma}, G) \to \mathrm{Epi}(\widehat{P}, G)$. The map $q \mapsto q \circ u$ described in the lemma completes the commutative square

$$\begin{array}{ccc} \mathrm{Epi}(\Gamma, G) & \longrightarrow & \mathrm{Epi}(P, G) \\ \uparrow & & \uparrow \\ \mathrm{Epi}(\widehat{\Gamma}, G) & \longrightarrow & \mathrm{Epi}(\widehat{P}, G) \end{array}$$

and hence is a bijection. $\square$

Proof of Proposition 3.2: If G is nilpotent of class c, and H is any group, then every homomorphism from H to G factors uniquely through H/H_c and hence there is a natural bijection $\mathrm{Epi}(H/H_c, G) \to \mathrm{Epi}(H, G)$. By combining two such bijections with the epimorphism $q \mapsto q \circ u$ of Lemma 3.4,

$$\mathrm{Epi}(\Gamma/\Gamma_c, G) \to \mathrm{Epi}(\Gamma, G) \to \mathrm{Epi}(P, G) \to \mathrm{Epi}(P/P_c, G),$$

we see that if G is finite, then $q \mapsto q \circ u_c$ defines a bijection $\mathrm{Epi}(\Gamma/\Gamma_c, G) \to \mathrm{Epi}(P/P_c, G)$.

Finitely generated nilpotent groups are residually finite, so for every $c > 0$ and every non-trivial element $\gamma \in P/P_c$, there is an epimorphism $\pi : P/P_c \to G$ to a finite (nilpotent) group such that $\pi(\gamma) \neq 1$. The preceding argument provides $q \in \mathrm{Epi}(\Gamma/\Gamma_c, G)$ such that $\pi = q \circ u_c$, whence $u_c(\gamma) \neq 1$. Thus u_c is injective.

u_c is also surjective, for if $\gamma \in \Gamma/\Gamma_c$ were not in the image then using the subgroup separability of nilpotent groups [33], we would have an epimorphism $q : \Gamma/\Gamma_c \to G$ to some finite group such that $q(\gamma) \notin q \circ u_c(P/P_c)$, contradicting the fact that $q \circ u_c$ is an epimorphism. $\square$

4. Finite Presentation: A Solution to Problem 1.1

Finitely generated free groups are residually torsion-free nilpotent [34], and hence so are subgroups of their direct products. Thus the following proposition resolves Problem 1.1.

Proposition 4.1. *If F is a finitely generated, non-abelian free group, then there exist finitely generated subgroups $P < F \times F$ such that the inclusion induces an isomorphism of pro-nilpotent completions, but P is not finitely presented.*

In what follows, we shall need to invoke the existence of finitely presented infinite groups Q with $H_2(Q, \mathbb{Z}) = 0$ that have no non-trivial finite quotients. A general method for constructing such groups is described in [14]. The first such group was discovered by Graham Higman:

$$Q = \langle a, b, c, d \mid bab^{-1} = a^2,\ cbc^{-1} = b^2,\ dcd^{-1} = c^2,\ ada^{-1} = d^2 \rangle.$$

The following proposition is a compilation of results due to Platonov and Tavgen [37] and Grunewald [24].

Proposition 4.2. *Let Q be a finitely presented infinite group with $H_2(Q, \mathbb{Z}) = 0$ that has no finite quotients. Let F be a finitely generated free group, let $F \to Q$ be a surjection, and let $P < F \times F$ be the associated fibre product. Then:*

1. *P is finitely generated but not finitely presented;*
2. *$(F \times F, P)$ is a Grothendieck pair.*

Proof. Let F be the free group on $\{x_1, \dots, x_n\}$. Since Q is finitely presented, the kernel of $F \to Q$ is the normal closure of a finite set $\{r_1, \dots, r_m\}$. It is easy to check that $P < F \times F$ is generated by $\{(x_1, x_1), \dots, (x_n, x_n), (r_1, 1), \dots, (r_m, 1)\}$. But Grunewald [24] proves that P is finitely presentable if and only if Q is finite. Assertion (2) is a special case of Proposition 3.1(1), and is due to Platonov and Tavgen [37]. □

5. Conjugacy Problem: A Solution to Problem 1.2

Recall that a finitely generated group G is said to have a *solvable conjugacy problem* if there is an algorithm that, given any pair of words in the generators, can correctly determine whether or not these words define conjugate elements of the group. If no such algorithm exists then one says that the group has an unsolvable conjugacy problem.

In the light of Proposition 3.2, the following theorem settles Problem 1.2. This theorem will be proved by combining the techniques of [12] with recent advances in the understanding of non-positively curved cube complexes and right-angled Artin groups (RAAGs). We remind the reader that a RAAG is a group with a finite presentation of the form

$$A = \langle a_1, \dots, a_n \mid [a_i, a_j] = 1 \forall (i, j) \in E \rangle.$$

Much is known about such groups. For our purposes here, their most important feature is that they are residually torsion-free-nilpotent ([20] Theorem 2.1).

Theorem 5.1. *There exist pairs of finitely presented, residually torsion-free-nilpotent groups $P \hookrightarrow \Gamma$ such that (Γ, P) is a Grothendieck Pair, but Γ has a solvable conjugacy problem while P has an unsolvable conjugacy problem.*

Proof. Let $1 \to N \to H \xrightarrow{p} Q \to 1$ be a short exact sequence of groups and let $P < H \times H$ be the associated fibre product. The 1-2-3 Theorem [6] states that if N is finitely generated, H is finitely presented, and Q has a classifying space with a finite 3-skeleton, then the P is finitely presented [6].

A further result from [6] states that if H is torsion-free and hyperbolic, and Q has an unsolvable word problem, then the fibre product $P < H \times H$ associated to any epimorphism will have an unsolvable conjugacy problem. On the other hand, the conjugacy problem in $H \times H$ is always solvable in polynomial time (cf. [12] §1.5).

Using ideas from [19], it is proved in [12] that there exist groups Q with no finite quotients that are of type F_3, have $H_2(Q, \mathbb{Z}) = 0$ and are such that the word problem in Q is unsolvable.

Combining the conclusion of the preceding three paragraphs, together with Proposition 3.1(1), we see that the theorem would be proved if, given a finitely presented group Q (as above), we could construct a short exact sequence $1 \to N \to H \to Q \to 1$ with N finitely generated and with H hyperbolic and residually torsion-free-nilpotent. For then $(H \times H, P)$ is the required pair of groups.

In [40] Rips describes an algorithm that, given a finite presentation of any group Q will construct a short exact sequence $1 \to K \to H \xrightarrow{p} Q \to 1$ where K is finitely generated and H is hyperbolic. There are many refinements of Rips's construction in the literature. Haglund and Wise [26] proved a version in which H is constructed to be *virtually special*. By definition, a virtually special group H has a subgroup of finite index $H_0 < H$ that is a subgroup of a RAAG [26], and as remarked upon above RAAG's are residually torsion-free-nilpotent by [20]. Consider the short exact sequence $1 \to K \cap H_0 \to H_0 \to p(H_0) \to 1$. If we take Q to be as in third paragraph of the proof, then $p(H_0) = Q$, because Q has no proper subgroups of finite index. And $N = K \cap H_0$, being of finite index in K, is finitely generated. Thus we have constructed a short exact sequence $1 \to N \to H_0 \to Q \to 1$ of the required form. □

Remarks 5.2. (1) In the preceding proof we quoted Haglund and Wise's version of the Rips construction; this was used to ensure that the group H was virtually special and hence

virtually residually torsion-free-nilpotent. Less directly, it follows from recent work of Agol [1] that the groups produced by the original Rips construction [40] are also virtually special.

(2) Whichever version of the Rips construction we use, we get a short exact sequence $1 \to N \to H \to Q \to 1$ with H hyperbolic (in particular finitely presented) and residually nilpotent, and with N finitely generated and not free. A further property of H is that it has cohomological dimension 2, and it is a theorem of Bieri [7] that a finitely presented normal subgroup of infinite index in a group of cohomological dimension 2 must be free. Thus if Q is infinite then N is not finitely presented. On the other hand, if Q has no finite quotients and $H_2(Q, \mathbb{Z}) = 0$, then (H, N) is a Grothendieck Pair. So, appealing once more to Proposition 3.2, we have further examples settling Problem 1.1.

(3) If one is willing to reduce the finiteness properties in the above theorem then one can simplify the proof: it is easy to prove that if Q has an unsolvable word problem and $p : H \to Q$ is an epimorphism from a hyperbolic group, then the conjugacy problem in ker p is unsolvable. So if $1 \to N \to H \to Q \to 1$ is constructed as in the above proof, then (H, N) is a Grothendieck pair of groups in the same nilpotent genus, but H is hyperbolic and N has an unsolvable conjugacy problem.

6. Nilpotent Genus and the Schur Multiplier: Problem 1.4

In the Introduction we explained how Baumslag [5] highlighted four questions in the theory of pro-nilpotent completions, the first of which we resolved by constructing a pair of finitely generated residually torsion-free-nilpotent groups of the same genus such that one was finitely presented and the other was not. Baumslag's second question is a homological variant on this: he asks if there is a pair of finitely generated residually torsion-free-nilpotent groups of the same genus such that the Schur multiplier $H_2(G, \mathbb{Z})$ of one is finitely generated, while that of the other group is not. We shall settle this question by proving the following theorem. Here, and in what follows, we take homology with coefficients in $\mathbb{Q}$. When we discuss the *dimension* of a homology group, we mean its dimension as $\mathbb{Q}$-vector space. Note that if $H_n(G, \mathbb{Z})$ is finitely generated then $H_n(G, \mathbb{Q})$ is finite dimensional.

Theorem 6.1. *There exists a pair of finitely generated residually torsion-free-nilpotent groups $N \hookrightarrow \Gamma$ that have the same nilpotent genus and the same profinite completion, but Γ is finitely presented while $H_2(N, \mathbb{Q})$ is infinite dimensional.*

Our proof of the above theorem draws on the ideas in previous sections, augmenting them with a spectral sequence argument to control the homology of N. The proof also relies on the construction of a group with particular properties that we regard as having independent interest – see Proposition 6.5.

6.1. **A spectral sequence argument.**

Proposition 6.2. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence. Suppose that N is finitely generated, that G is finitely presented and $H_3(G, \mathbb{Q})$ is finite dimensional, and that Q is finitely presented but $H_3(Q, \mathbb{Q})$ is infinite dimensional. Then $H_2(N, \mathbb{Q})$ is infinite dimensional.*

Proof. The Lyndon-Hochschild-Serre (LHS) spectral sequence calculates the homology of G in terms of N and Q. The terms on the E^2 page of the spectral sequence are $E^2_{pq} = H_p(Q, H_q(N, \mathbb{Q}))$, where the action of Q on $H_*(N, \mathbb{Q})$ is induced by the action of G on N by conjugation. (The action of G on $H_*(N, \mathbb{Q})$ gives rise to a Q-action because the action of N on itself by conjugation induces the trivial action on homology.)

Our finiteness assumptions on N, G and Q imply that the entries E^2_{pq} are finite dimensional for $0 \leq p \leq 2$ and $0 \leq q \leq 1$ but that $E^2_{30} = H_3(Q, \mathbb{Q})$ is infinite dimensional. The effect of the differentials $d_2 : E^2 \to E^2$ does not alter the (in)finite dimensionality of these terms. Likewise $E^3_{02} = H_0(Q, H_2(N, \mathbb{Q}))/d_2 E^2_{21}$ is finite dimensional if and only if $H_0(Q, H_2(N, \mathbb{Q}))$ is finite dimensional.

The kernel of $d_3 : E^3_{30} \to E^3_{02}$ is E^∞_{30}, which is a section of $H_3(G,\mathbb{Q})$ and hence is finite dimensional. So the image of the map to E^3_{02} is infinite dimensional. But E^3_{02} is a quotient of $H_0(Q, H_2(N,\mathbb{Q}))$, which in turn is a quotient of $H_2(N,\mathbb{Q})$. Thus $H_2(N,\mathbb{Q})$ is infinite dimensional. □

6.2. **A designer group.** Recall that a group G is termed *acyclic* (over $\mathbb{Z}$) if $H_i(G,\mathbb{Z}) = 0$ for all $i \geq 1$. The Higman group described in Section 4 was the first example of a finitely presented acyclic group with no proper subgroups of finite index. Further examples were constructed in [14], including, for each integer $p \geq 3$,

$$\langle a_1, a_2, b_1, b_2 \mid a_1^{-1}a_2^p a_1 a_2^{-p-1},\ b_1^{-1}b_2^p b_1 b_2^{-p-1},\ a_1^{-1}[b_2, b_1^{-1}b_2b_1],\ b_1^{-1}[a_2, a_1^{-1}a_2a_1]\rangle.$$

Let A be one of the above groups. The salient features of A are that it is finitely presented, acyclic over $\mathbb{Z}$, has no finite quotients, contains a 2-generator free group, F say, and is torsion-free (indeed it has a 2-dimensional classifying space $K(A,1)$, cf. [14] p.364). Let $\Delta = (A\times A) *_S (A\times A)$ be the double of $A\times A$ along $S < F\times F$, where S is the first Stallings-Bieri group, i.e. the kernel of a homomorphism $F\times F \to \mathbb{Z}$ whose restriction to each of the factors is surjective. The key features of S are that it is finitely generated but $H_2(S,\mathbb{Q})$ is infinite dimensional (see [41], or [15] pp. 482-485).

Lemma 6.3. *Δ is torsion-free, finitely presented, has no non-trivial finite quotients, and $H_3(\Delta,\mathbb{Q})$ is infinite dimensional.*

Proof. The amalgamated free product of two finitely presented groups along a finitely generated subgroup is finitely presented, so Δ is finitely presented. And an amalgam of torsion-free groups is torsion-free. The four visible copies of A generate Δ, and these all have trivial image in any finite quotient, so Δ has no non-trivial finite quotients. We calculate $H_3(\Delta,\mathbb{Q})$ using the Mayer-Vietoris sequence (omitting the coefficient module $\mathbb{Q}$ from the notation):

$$\dots H_3(A\times A)\oplus H_3(A\times A) \to H_3(\Delta) \to H_2(S) \to H_2(A\times A)\oplus H_2(A\times A) \to \dots$$

As A is acyclic, so is $A\times A$, by the Künneth formula. Hence $H_3(\Delta,\mathbb{Q}) \cong H_2(S,\mathbb{Q})$ is infinite dimensional. □

Recall that a group G is termed *super-perfect* if $H_1(G,\mathbb{Z}) = H_2(G,\mathbb{Z}) = 0$. Proposition 3.1 explains our interest in this condition. Δ is perfect but it is not super-perfect.

Lemma 6.4. *$H_2(\Delta,\mathbb{Z}) \cong H_1(S,\mathbb{Z}) \cong \mathbb{Z}^3$.*

Proof. A slight variant of the above Mayer-Vietoris argument shows that $H_2(\Delta,\mathbb{Z}) \cong H_1(S,\mathbb{Z})$.

The first homology of S can be calculated using Theorem A of [17]: if $G \leq F\times F$ is a subdirect product and we write $F_1 = F\times 1$ and $L = G\cap F_1$, then

$$H_1(G,\mathbb{Z}) \cong H_1(F,\mathbb{Z}) \oplus H_2(F_1/L,\mathbb{Z}) \oplus C,$$

where $C = \ker(H_1(F_1,\mathbb{Z}) \to H_1(F_1/L,\mathbb{Z}))$.

In our case, $G = S$ and $F_1/L = \mathbb{Z}$, so $H_2(F_1/L,\mathbb{Z}) = 0$ and C is cyclic. (Recall that $F \cong F_1$ is free of rank 2.) □

To obtain a super-perfect group, we pass to the *universal central extension* $\widetilde{\Delta}$.

Proposition 6.5. *$\tilde{\Delta}$ is torsion-free, finitely presented, super-perfect, has no non-trivial finite quotients, and $H_3(\tilde{\Delta},\mathbb{Q})$ is infinite dimensional.*

Proof. The standard theory of universal central extensions (see [35], Chapter 5) tells us that $\tilde{\Delta}$ is perfect and that there is a short exact sequence

$$1 \to H_2(\Delta,\mathbb{Z}) \to \tilde{\Delta} \to \Delta \to 1.$$

Since Δ and $H_2(\Delta,\mathbb{Z})$ are finitely presented and torsion-free, so is $\tilde{\Delta}$. Since Δ has no non-trivial finite quotients, $H_2(\Delta,\mathbb{Z})$ would have to map onto any finite quotient of $\tilde{\Delta}$, which means that all

such quotients are abelian. Since $\tilde{\Delta}$ is perfect, it follows that it has no non-trivial finite quotients (cf. [14], p.369).

To see that $H_3(\tilde{\Delta}, \mathbb{Q})$ is infinite dimensional, we consider the LHS spectral sequence associated to the above short exact sequence. As Δ is finitely presented and $K := H_2(\Delta, \mathbb{Z})$ is of type FP_∞, all of the groups in the first three columns, $E^2_{pq} = H_p(\Delta, H_q(K, \mathbb{Q}))$ with $0 \le p \le 2$, are finite dimensional. On the other hand, $E^2_{30} = H_3(\Delta, \mathbb{Q})$ is infinite dimensional. Therefore $E^3_{30} = \ker(\mathrm{E}^2_{30} \to \mathrm{E}^2_{11})$ and $E^\infty_{30} = E^4_{30} = \ker(\mathrm{E}^3_{30} \to \mathrm{E}^3_{02})$ are infinite dimensional. But E^∞_{30} is a quotient of $H_3(\tilde{\Delta}, \mathbb{Q})$, so $H_3(\tilde{\Delta}, \mathbb{Q})$ is also infinite dimensional. $\square$

6.3. **Proof of Theorem 6.1.** We have constructed a group $\tilde{\Delta}$ that is super-perfect, finitely presented and has $H_3(\tilde{\Delta}, \mathbb{Q})$ infinite dimensional. By applying a suitable version of the Rips construction to $\tilde{\Delta}$ (as in the proof of Theorem 5.1), we obtain a short exact sequence

$$1 \to N \to H \to \tilde{\Delta} \to 1$$

with N finitely generated and H is a 2-dimensional hyperbolic group that is virtually special. Passing to a subgroup of finite index $H_0 < H$ and replacing N by $N \cap H_0$, we may assume that H is a subgroup of a RAAG, and hence is residually torsion-free-nilpotent. Proposition 3.1(2) tells us that (H, N) is a Grothendieck Pair, so by Proposition 3.2 H and N have the same nilpotent genus. Proposition 6.2 tells us that $H_2(N, \mathbb{Q})$ is infinite dimensional. $\square$

7. Normal subgroups of parafree groups

We settled Baumslag's first three questions by constructing groups of a somewhat pathological nature that lie in the same nilpotent genus as well behaved groups. Our solution to Problem 1.4 is of an entirely different nature: the point here is to prove that parafree groups share a significant property with free groups. Correspondingly, the nature of the mathematics that we shall draw on is entirely different.

Our proof of the following theorem relies on a mix of L^2 Betti numbers and profinite group theory that we first employed in our paper [13] with M. Conder.

Theorem 7.1. *(=Theorem C) If Γ is a finitely generated parafree group, then every finitely generated non-trivial normal subgroup of Γ is of finite index.*

7.1. L^2 **Betti numbers.** The standard reference for this material is Lück's treatise [31]. In what follows $b_1(X)$ denotes usual first Betti number of a group, and $b_1^{(2)}$ denotes the first L^2 Betti number. We shall not recall the definition of the L^2 Betti number as it does not inform our arguments. In the case of finitely presented groups, one can use *Lück's Approximation Theorem* [30] to give a surrogate definition of $b_1^{(2)}$: Suppose that Γ is *finitely presented* and let

$$\Gamma = N_1 > N_2 > \ldots > N_m > \ldots,$$

be a sequence of normal subgroups, each of finite index in Γ, with $\bigcap_m N_m = 1$; Lück proves that

$$\lim_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} = b_1^{(2)}(\Gamma).$$

Example 7.2. Let F be a free group of rank r. Euler characteristic tells us that a subgroup of index d in F is free of rank $d(r-1)+1$, so by Lück's Theorem $b_1^{(2)}(F_r) = r-1$. A similar calculation shows that if Σ is the fundamental group of a closed surface of genus g, then $b_1^{(2)}(\Sigma) = 2g-2$.

If one assumes only that the group Γ is *finitely generated*, then one does not know if the above limit exists, and when it does exist one does not know if it is independent of the chosen tower of subgroups. This is a problem in the context of Theorem 7.1 because we do not know if finitely generated parafree groups are finitely presentable. Thus we appeal instead to the weaker form of Lück's approximation theorem established for finitely generated groups be Lück and Osin [32].

Theorem 7.3. *If Γ is a finitely generated residually finite group and (N_m) is a sequence of finite-index normal subgroups with $\bigcap_m N_m = 1$, then*

$$\limsup_{m\to\infty} \frac{b_1(N_m)}{[\Gamma : N_m]} \leq b_1^{(2)}(\Gamma).$$

Another result about L^2 Betti numbers that we will make use of is the following theorem of Gaboriau (see [21] Theorem 6.8).

Theorem 7.4. *Suppose that*

$$1 \to N \to \Gamma \to \Lambda \to 1$$

is an exact sequence of groups where N and Λ are infinite. If $b_1^{(2)}(N) < \infty$, then $b_1^{(2)}(\Gamma) = 0$.

7.2. L^2 **Betti numbers of dense subgroups.** The key step in the proof of Theorem 7.1 is the following result (cf. [13] Proposition 3.2). Recall that we write $\widehat{\Gamma}_p$ to denote the pro-p completion of a group Γ, and $\overline{H}$ to denote the closure of a subgroup in $\widehat{\Gamma}_p$.

Proposition 7.5. *Let Γ be finitely generated group and let F be a finitely presented group that is residually-p for some prime p. Suppose that there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ and that $\overline{\Gamma} = \widehat{F}_p$. Then $b_1^{(2)}(\Gamma) \geq b_1^{(2)}(F)$.*

Proof. For each positive integer d let $N(d) < F$ be the intersection of all normal subgroups of p-power index at most d in F. $L(d) = \Gamma \cap \overline{N(d)} < \widehat{F}_p$. We saw in Corollary 2.6 that $\bigcap_d \overline{N}(d) = 1$, hence $\bigcap_d L(d) = 1$. Since Γ and F are both dense in $\widehat{F}_p$, the restriction of $\widehat{F}_p \to \widehat{F}_p/\overline{N(d)}$ to each of these subgroups is surjective, and therefore

$$[\Gamma : L(d)] = [\widehat{F}_p : \overline{N(d)}] = [F : N(d)].$$

$L(d)$ is dense in $\overline{N(d)}$ and in Lemma 2.8 we saw that $\widehat{N(d)}_p \cong \overline{N(d)}$, so Lemma 2.9 implies that $b_1(L(d)) \geq b_1(N(d))$. We use the towers $(L(d))$ in Γ and $(N(d))$ in F to compare L^2-betti numbers, applying Theorem 7.3 to the finitely generated group Γ and Lück's Approximation Theorem to the finitely presented group Γ:

$$b_1^{(2)}(\Gamma) \geq \limsup_{d\to\infty} \frac{b_1(L(d))}{[\Gamma : L(d)]} \geq \limsup_{d\to\infty} \frac{b_1(N(d))}{[F : N(d)]} = \lim_{d\to\infty} \frac{b_1(N(d))}{[F : N(d)]} = b_1^{(2)}(F)$$

as required. □

Corollary 7.6. *Let Λ and Γ be finitely presented groups that are residually-p for some prime p. If $\widehat{\Gamma}_p \cong \widehat{\Lambda}_p$ then $b_1^{(2)}(\Gamma_1) = b_1^{(2)}(\Gamma_2)$.*

By combining this proposition with Theorem 7.4 we deduce:

Theorem 7.7. *Let Γ be a finitely generated group and let $N \lhd \Gamma$ be a non-trivial normal subgroup. Let F be a finitely presented group that is residually-p for some prime p and suppose that there is an injection $\Gamma \hookrightarrow \widehat{F}_p$ with dense image. If $b_1^{(2)}(F) > 0$, then either $b_1^{(2)}(N) = \infty$ or else $|\Gamma/N| < \infty$. In particular, if N is finitely generated then it is of finite index.*

Proof of Theorem 7.1. Suppose now that Γ is a finitely generated parafree group, with the same nilpotent genus as the free group F, say. Finitely generated groups with the same nilpotent genus have isomorphic pro-p completions for every prime p (cf. Remark 2.1(9)), so $\widehat{\Gamma}_p \cong \widehat{F}_p$.

If F is cyclic, then it is easy to see that Γ must also be cyclic, so the conclusion of Theorem 7.1 holds. So we assume that F has rank $r > 1$.

Γ is residually-p for all primes p. Indeed, since Γ is residually nilpotent, every non-trivial element of Γ has a non-trivial image in $\Gamma/\Gamma_c \cong F/F_c$ for some term Γ_c of the lower central series, and the

free nilpotent group F/F_c is residually-p for all primes p (see [23]). Combining this observation with the conclusion of the first paragraph, we see that the natural map from Γ to $\widehat{\Gamma}_p \cong \widehat{F}_p$ is injective. In Example 7.2 we showed that $b_1^{(2)} = (r-1) > 0$. Thus we are in the situation of Theorem 7.7 and the proof is complete. □

8. Final Comments

We close with some further consequences of the arguments in the preceding section, and a discussion of related matters.

8.1. **Parafree groups and lattices.** Recall that a group Γ is said to be *parafree of rank* r if Γ is residually nilpotent and is in the same nilpotent genus as a free group of rank r. (It follows from the residual nilpotence of free groups [34] that an r-generator parafree group of rank r is free of rank r.) As a special case of Corollary 7.6 we have:

Corollary 8.1. *Let Γ be a finitely presented parafree group of rank r. Then $b_1^{(2)}(\Gamma) = r - 1 = b_1(\Gamma) - 1$.*

From this we deduce:

Corollary 8.2. *Let Γ be a finitely presented parafree group of rank $r \geq 2$ which is not free. Then Γ is not isomorphic to a lattice in a connected Lie group.*

Proof. From Corollary 8.1, $b_1^{(2)}(\Gamma) \neq 0$. It is shown in [28] that if Γ is a lattice in a connected Lie group with $b_1^{(2)}(\Gamma) \neq 0$, then Γ is commensurable with a lattice in $\mathrm{PSL}(2,\mathbb{R})$; i.e. the group is virtually free or virtually the fundamental group of a closed orientable surface of genus at least 2. Now Γ is torsion-free, with torsion-free abelianization, so in fact, in this case, Γ is free or the fundamental group Σ_g of a closed orientable surface of genus $g \geq 2$. The former is ruled out by assumption, and the latter is ruled out by the observation that $b_1^{(2)}(\Sigma_g) = 2g - 2 = b_1(\Sigma_g) - 2$; see Example 7.2. (Alternatively, it is straightforward to construct a finite nilpotent quotient of a free group of rank $2g$ that cannot be a quotient of the genus g surface group.) □

8.2. **Homology boundary links.** In contrast to Corollary 8.2, we shall see that there are lattices in $\mathrm{PSL}(2,\mathbb{C})$ that *do* have the same lower central series as a free group.

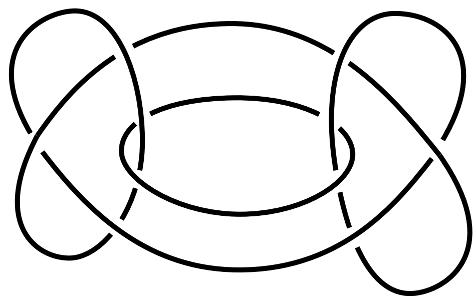
Example 8.3. Recall that a link $L \subset S^3$ of $m \geq 2$ components is called a *homology boundary link* if there exists an epimorphism $h : \pi_1(S^3 \setminus L) \to F$ where F is a free group of rank m. By Alexander duality, $H_1(\pi_1(S^3 \smallsetminus L), \mathbb{Z}) \cong \mathbb{Z}^m$. Thus h induces an isomorphism $H_1(\pi_1(S^3 \smallsetminus L), \mathbb{Z}) \cong H_1(F_m, \mathbb{Z})$. For trivial reasons, h induces also induces a surjection $H_k(\pi_1(S^3 \smallsetminus L), \mathbb{Z}) \to H_k(F, \mathbb{Z})$ for all $k \geq 2$. Therefore, by Stallings' Theorem [42], h induces an isomorphism between the lower central series quotients of $\pi_1(S^3 \smallsetminus L)$ and F.

Let L be a homology boundary link. If $S^3 \smallsetminus L \cong \mathbb{H}^3/\Gamma$ with $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ a lattice, then $b_1^{(2)}(\Gamma) = 0$, by [28]. In the light of this, we have the following consequence of Corollary 7.6.

Corollary 8.4. *Suppose that L is a homology boundary link and $S^3 \smallsetminus L \cong \mathbb{H}^3/\Gamma$, with Γ a lattice. Then, Γ is not residually-p for any prime p. In particular Γ is not residually nilpotent.*

Standard results about lattices in $\mathrm{PSL}(2,\mathbb{C})$ guarantee that they are *virtually* residually p for all but a finite number of primes p (and hence virtually residually nilpotent). Moreover, recent work of Agol [1] implies that they are virtually residually torsion-free-nilpotent.

Example 8.5. We shall verify that the link L shown below is a homology boundary link. The complement of this link is hyperbolic, $S^3 \smallsetminus L \cong \mathbb{H}^3/\Gamma$ where $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is the torsion-free non-uniform lattice denoted A2 in [18].



To see that Γ surjects the free group of rank 2, we argue that 0-surgery on both components results in a connect sum of two copies of $\mathbb{S}^2 \times \mathbb{S}^1$. We give the details at the level of the fundamental group.

From [18], we have the presentation

$$\Gamma = \langle u, v, z \mid [u, l] = 1, uzu^{-1} = v^{-1}zvz, l = v^{-1}uzu^{-1}vz \rangle,$$

where u is a meridian for the unknotted component and l is a longitude for u. The other peripheral subgroup (i.e. the one corresponding to the square knot component) is $\langle uzu^{-1}, uv^2uv \rangle$, with meridian uv^2uv and longitude uzu^{-1}.

Now, performing 0-surgery on the unknotted component trivializes l, so in the image of π_1 of the complement, the first relation in our presentation becomes redundant. Performing 0-surgery on the second component imposes the relation $uzu^{-1} = 1$, hence $z = 1$, and the remaining relations in our presentation become redundant. Thus the fundamental group of the surgered manifold is simply $\langle u, v \mid \rangle$, as required. $\square$

8.3. **Pro-p goodness.** Homology boundary links also provide some interesting examples in the following related context.

One says that a group Γ is *pro-p good* if the homomorphism of cohomology groups

$$H^n(\widehat{\Gamma}_p; \mathbb{F}_p) \to H^n(\Gamma; \mathbb{F}_p)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism, where the group on the left is in the continuous cohomology of $\widehat{\Gamma}$. One says that the group Γ is *cohomologically complete* if Γ is pro-p good for all primes p.

It is shown in [9] that many link groups are cohomologically complete, and indeed it was claimed by Hillman, Matei and Morishita [27] that all link groups are cohomologically complete. Counterexamples were given in [8] (the examples are split links). Here we note that homology boundary links provide other counterexamples. In particular there are hyperbolic links that provide counterexamples.

Proposition 8.6. *Let L be a homology boundary link. Then $\pi_1(S^3 \smallsetminus L)$ is not pro-p good for any prime p. In particular, $\pi_1(S^3 \smallsetminus L)$ is not cohomologically complete.*

Proof. Let $\Gamma = \pi_1(S^3 \smallsetminus L)$. We saw in Example 8.3 that $\widehat{\Gamma}_p$ is isomorphic to a free pro-p group. Hence $H^2(\widehat{\Gamma}_p; \mathbb{F}_p) = 0$. On the other hand, since L is a link with $m \geq 2$ components, $H^2(\Gamma; \mathbb{F}_p)$ has dimension $m - 1 > 0$ as an $\mathbb{F}_p$-vector space. $\square$

By way of contrast, we also note that the lattice Γ described in §8.2 is *good* in the sense of Serre; i.e. for every finite Γ-module M, the homomorphism of cohomology groups

$$H^n(\widehat{\Gamma}; M) \to H^n(\Gamma; M)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$. Goodness of Γ follows from [25] since Γ is a subgroup of finite index in a Bianchi group.

8.4. Theorem 7.7 can be usefully applied to cases where F is not free; for example, F might be a non-abelian surface group or more generally non-abelian limit group. That these satisfy the condition on the first L^2 Betti number can be seen in Example 7.2 for surface groups and [16] for limit groups. This leads to an analogue of Theorem 7.1 for *paralimit* groups. We remark that the notion of "para-surface group" was considered in [10].

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